Homework 7 Sample Solutions

Exercise 8.1 #14. Solve $\frac{dx}{dt} = 1 - 3x$, where $x(-1) = -2$.

Solution. We separate variables and integrate to obtain

$$\int \frac{dx}{1-3x} = \int dt = t + C$$

To compute the lefthand integral, we use u substitution, with $u = 1 - 3x$, $du = -3\,dx$:

$$\int \frac{dx}{1-3x} = -\frac{1}{3}\int \frac{du}{u} = -\frac{1}{3}\ln|u| = -\frac{1}{3}\ln|1-3x| = t + C$$

Now we multiply both sides by -3 and exponentiate both sides:

$$\ln|1-3x| = -3t + C_1$$

$$|1-3x| = e^{-3t+C_1} = e^{C_1}e^{-3t} = C_2e^{-3t}$$

(I use C_1, C_2, etc. whenever a new constant is introduced. The important thing to remember is that every C_i is an *arbitrary* constant.) Now $|1-3x|$ is $\pm(1-3x)$, where the sign depends on the value of x. However, I can just incorporate this in the constant on the other side:

$$\pm(1-3x) = C_2e^{-3t}$$

$$1 - 3x = \pm C_2e^{-3t} = C_3e^{-3t}$$

$$x = \frac{1}{3}(1 - C_3e^{-3t}) = \frac{1}{3} + C_4e^{-3t}$$

Now we plug in the initial condition to find the value of the constant:

$$x(-1) = -2 = \frac{1}{3} + C_4e^{-3(-1)} = \frac{1}{3} + C_4e^3$$

$$-\frac{7}{3} = C_4e^3$$

$$C_4 = -\frac{7}{3}e^{-3}$$

Thus the solution is

$$x(t) = \frac{1}{3} - \frac{7}{3}e^{-3t-3}$$

□

Exercise 8.1 #22. Denote by $L(t)$ the length of a fish at time t, and assume that the fish grows according to the von Bertalanffy equation

$$\frac{dL}{dt} = k(34 - L(t)) \text{ with } L(0) = 2$$

(a) Solve the differential equation.

(b) Use your solution in (a) to determine k under the assumption that $L(4) = 10$. Sketch the graph of $L(t)$ for this value of k.

(c) Find the length of the fish when $t = 10$.

(d) Find the asymptotic length of the fish; that is, find $\lim_{t\to\infty} L(t)$.

Solution. (a) We follow the same steps as in problem 14:

$$\int \frac{dL}{34 - L} = \int k dt = kt + C$$

$$\begin{aligned}
\int \frac{dL}{34 - L} &= -\ln|34 - L| = kt + C \\
\ln|34 - L| &= -kt + C_1 \\
|34 - L| &= e^{-kt+C_1} = C_2 e^{-kt} \\
34 - L &= \pm C_2 e^{-kt} = C_3 e^{-kt} \\
L(t) &= 34 - C_3 e^{-kt}
\end{aligned}$$

Plugging in the initial condition:

$$L(0) = 2 = 34 - C_3 e^{-k(0)} = 34 - C_3 e^0 = 34 - C_3$$

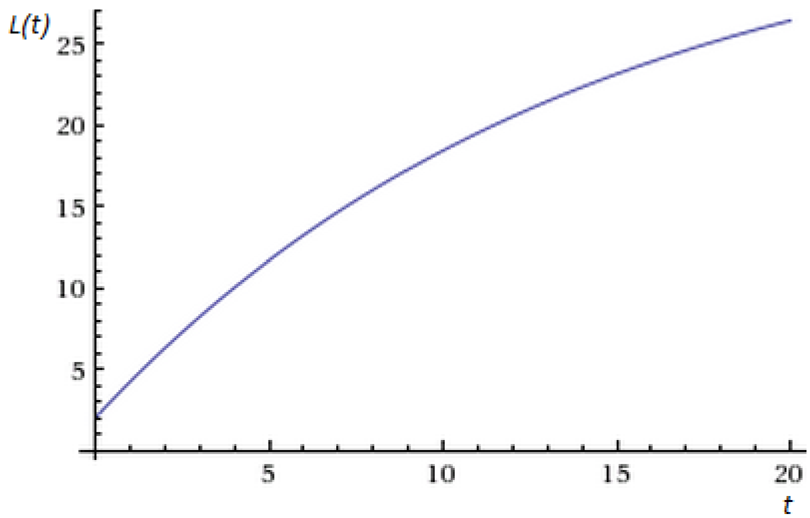
$$\begin{aligned}
C_3 &= 32 \\
L(t) &= 34 - 32e^{-kt}
\end{aligned}$$

(b) We just plug this condition in, as if it were another initial condition:

$$L(4) = 10 = 34 - 32e^{-k(4)} = 34 - 32e^{-4k}$$

$$\begin{aligned}
32e^{-4k} &= 24 \\
e^{-4k} &= \frac{24}{32} = \frac{3}{4} \\
-4k &= \ln\frac{3}{4} \\
k &= \frac{1}{4}\ln\frac{4}{3}
\end{aligned}$$

Here's my sketch of $L(t)$ with this value of k:



(c) Simply plug in $t = 10$:

$$L(10) = 34 - 32e^{-1/4(\ln 4/3)10} = 34 - 32(e^{\ln 4/3})^{-10/4} = 34 - 32(\frac{4}{3})^{-5/2}$$

$$= 34 - 32(\frac{3}{4})^{5/2} \approx 18.41$$

(d) We are computing the limit

$$\lim_{t\to\infty} 34 - 32e^{-1/4(\ln 4/3)t}$$

To figure this out, we need to know whether $-\frac{1}{4}\ln\frac{4}{3}$ is positive or negative. Since $\frac{4}{3} > 1$, $\ln\frac{4}{3} > \ln 1 = 0$, so it turns out that $-\frac{1}{4}\ln\frac{4}{3}$ is negative. Therefore

$$\lim_{t\to\infty} 34 - 32e^{-1/4(\ln 4/3)t} = 34 - 32(0) = 34$$

so the asymptotic length of the fish is 34.

□

Exercise 8.1 #48. Solve the equation $\frac{dy}{dx} = x^2y^2$, with $y_0 = 1$ if $x_0 = 1$.

Solution. Separate variables and integrate as usual:

$$\int \frac{dy}{y^2} = \int x^2 dx$$

$$-\frac{1}{y} = \frac{1}{3}x^3 + C$$

Solving for y,

$$y = \frac{-1}{\frac{1}{3}x^3 + C} = \frac{-3}{x^3 + C_1}$$

Plugging in the initial condition,

$$y_0 = 1 = \frac{-3}{1^3 + C_1} = \frac{-3}{1 + C_1}$$
$$1 + C_1 = -3$$
$$C_1 = -4$$

Thus our solution is

$$y = \frac{-3}{x^3 - 4}$$

□

Exercise 8.1 #54. Consider the following differential equation, which is important in population genetics:

$$a(x)g(x) - \frac{1}{2}\frac{d}{dx}[b(x)g(x)] = 0$$

Here, $b(x) > 0$.

(a) Define $y = b(x)g(x)$, and show that y satisfies

$$\frac{a(x)}{b(x)}y - \frac{1}{2}\frac{dy}{dx} = 0$$

(b) Separate variables in the above equation and show that if $y > 0$, then

$$y = C \exp\left[2 \int \frac{a(x)}{b(x)}\, dx\right]$$

Solution. (a) Setting $y = b(x)g(x)$, we have that $g(x) = \frac{y}{b(x)}$ since $b(x) > 0$. Thus, we can make this substitution to get

$$a(x)g(x) - \frac{1}{2}\frac{d}{dx}[b(x)g(x)] = a(x)\frac{y}{b(x)} - \frac{1}{2}\frac{d}{dx}\left[b(x)\frac{y}{b(x)}\right] = \frac{a(x)}{b(x)}y - \frac{1}{2}\frac{dy}{dx} = 0$$

which is what we wanted to show.

(b) Separating variables:

$$\frac{a(x)}{b(x)}y = \frac{1}{2}\frac{dy}{dx}$$
$$2\int \frac{a(x)}{b(x)}\, dx = \int \frac{dy}{y} = \ln|y| + C_1$$

Since we are assuming that $y > 0$, this is

$$\ln y + C_1 = 2\int \frac{a(x)}{b(x)}\, dx$$

$$\ln y = -C_1 + 2\int \frac{a(x)}{b(x)}\,dx$$

$$y = \exp\left[-C_1 + 2\int \frac{a(x)}{b(x)}\,dx\right] = C\exp\left[2\int \frac{a(x)}{b(x)}\,dx\right]$$

which is what we wanted to show.

Aside: It is not necessary to assume that $y > 0$ in this problem. If $y < 0$, then we would simply get a negative value of C instead of a positive one. If $y = 0$, then this equation still holds, but with $C = 0$. So in fact, no assumption on y is necessary at all. □

Exercise 9.1 #5. Determine c such that

$$\begin{aligned} 2x - 3y &= 5 \\ 4x - 6y &= c \end{aligned}$$

has **(a)** infinitely many solutions and **(b)** no solutions. **(c)** Is it possible to choose a number for c so that the system has exactly one solution? Explain your answer.

Solution. Let's begin by solving the system as much as we can. Subtracting 2 times the upper equation from the bottom equation we get

$$\begin{aligned} 2x - 3y &= 5 \\ 0 &= c - 10 \end{aligned}$$

Thus, if $c - 10$ is not actually 0, then this system is inconsistent. That is to say, the system has no solutions if $c \neq 10$ this answers **(b)**.

On the other hand, if $c = 10$, then our system is

$$\begin{aligned} 2x - 3y &= 5 \\ 0 &= 0 \end{aligned}$$

It follows that this system has infinitely many solutions, so **(a)** holds exactly when $c = 10$.

Note that we have considered all possibilities: Either $c = 10$ or $c \neq 10$. In the former case, there are infinitely many solutions, and in the latter case, there are no solutions. There is no other possibility, so in particular, there is no value of c such that the system has exactly one solution. This answers **(c)**. □